Extending Kepler's *Mysterium Cosmographicum*

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Abstract

An attempt is made to extend Kepler's classic model of the solar system to account for Uranus and Neptune. Possible additions to the five Platonic solids are examined, with the choice finally being made to use a d10 (pentagonal trapezohedron) and a Utah Teapot. The relative orbit sizes thus produced are compared to modern figures and Kepler's original results.

I. Introduction

Mysterium Cosmographicum was published by Johannes Kepler in 1596. In it, the relative spacing of the orbits of the six planets then known (Mercury, Venus, Earth, Mars, Jupiter, Saturn) were explained in terms of a nested series of Platonic solids (octohedron, icosahedron, dodecahedron, tetrahedron, cube).[Kepler, 1596] While the fit was far from perfect, Kepler never entirely dropped the concept.[Kepler, 1618]

Not actually being nessecary to a scientific understanding of the solar system, Kepler's *Mysterium Cosmographicum* model did not prove nearly as influential as his laws of planetary motion. Outside of the scientific world, however, his imagery and mysticism did have a lasting impact.

In order to explore this strange, beautiful, flawed model, the author set about extending it to account for Uranus and Neptune. It was hoped that this effort would help modern readers appreciate some of the significant differences in the scientific mindset of Kepler's age.

II. Methods

There are two components to the Keplerian model: the spherical shells representing the orbits of the planets, and the Platonic solids between them which provide the spacing. The spherical shells have a thickness based on the eccentricity of that planet's orbit. Thus, Mercury and Mars each have a relatively thick shell, while those for Venus and Earth are much thinner.

i. Selection of solids

One major reason Kepler was a proponent of the *Mysterium Cosmographicum* model was the apparent numerological significance of six planets with five Platonic solids to provide spacing between them. This poses a particular challenge when trying to extend the model to another two planets, as a total of seven Platonic solids would be needed.

One option would be to explore higher dimensional spaces. In four dimensions, for instance, there are six regular polytopes. However, in all dimensions higher than four there are only three regular polytopes.[Coxeter, 1973] Thus, even if one were to find the use of higher dimensional objects in this context acceptable, it isn't clear which two should be chosen. This option was rejected as being contrary to the numerological biases of the original model.

Without real Platonic solids available, the author was forced to approach the problem more

Figure 1: *The Six Platonic Solids, James Arvo and David Kirk, 1987 [Avro, 1987]*

abstractly. Luckily, two options immediately presented themselves.

Utah teapot

The Utah teapot is a classic computer graphics object dating back to 1975. It has been included as a pre-defined primitive in basically every graphics package ever since. Having a simple object with relatively complicated geometry has proved to be very valuable, and the Utah teapot has become the default for that role. Because of its ubiquitous inclusion amongst other basic solids, it has often been referred to as the "sixth Platonic solid".1 The author felt this amply justified its use here.

d10

To the best of the author's knowledge, the only context in real life where all the Platonic solids are commonly seen is in a set of gaming dice. These normally include a tetrahedron (d4), cube (d6), octahedron (d8), dodecahedron (d12), and icosahedron (d20). They also include a pentagonal trapezohedron (d10).2 While a d10 is not technically a Platonic solid, being part of a group otherwise made up of Platonic solids makes it good candidate for the seventh.

Figure 2: *A set of gaming dice*

ii. Modeling

The 3D modelling was all performed in Blender, a GPLv2 licensed free software package. Sizing of the elements was performed manually, using the renderer itself to check for collisions between the objects.

The model was built starting with the shell of Jupiter being set to an arbitrary diameter of 10 units. Its thickness was set based on modern number for Jupiter's perihelion (4.95029 AU) and aphelion (5.45492 AU)[Simon, 1994], for a percentage of 9.241%. From this base, the next solid outward (cube) and inward (tetrahedron) were placed, and scaled to just touch the shell of Jupiter. The shells for Saturn and Mars were added and their thickness set, again using the modern perihelial and aphelial measurements. This was repeated for the entire system. For the Utah teapot, the shell was fitted only to the rounded belly of the teapot. The handle and spout were ignored.3

III. Results

After building up the model, the relative sizes of the orbits could be compared. It is important to remember that relative sizes is all that Kepler was interested in, as absolute measurements weren't possible yet. (This is still reflected in

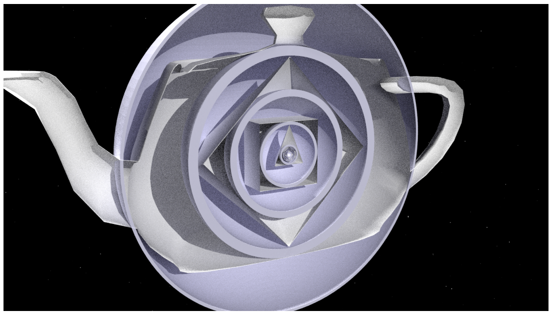

Figure 3: *The extended Keplerian model*

the use of the AU, or astronomical unit. Defining that unit as the size of the Earth's orbit wasn't an act of supreme cosmological arrogance, it was simply using the only yardstick available.) For each pair of neighboring planets, the percentage size of the outer orbit in terms of the inner orbit is given, for both the mean and aphelial diameters.1

While the fit isn't very good by modern standards, the addition of Uranus and Neptune doesn't stand out as particularly terrible compared to Kepler's original results. The fit for Uranus-Neptune, in fact, using the Utah teapot solid, provides one of the most accurate fits of all.2

It is clear that the d10 is a particularly poor fit for the Saturn-Uranus pair. It can at least be said that the result here when looking at the aphelial ratios is still better than Kepler's octohedron between Mercury and Venus. This could be seen a cherry-picking, but the author feels it is well within the spirit of Kepler's original work, which freely switched between perihelial, mean and aphelial numbers as convenient. That said, further research is needed to find a more appropriate Platonic solid alternative.

IV. Discussion

An extended *Mysterium Cosmographicum* model of the solar system was never going to provide new scientific insights, but it does have value in exploring the mindset of the 17th century.

It can be challenging for the modern reader to think in their terms, particularly the bias towards using ratios and proportions instead of absolute measurements. This was the standard until well after Kepler, largely because the metrology simply didn't exist yet to support absolute measurements of any quality. In addition, geometry was still the predominant mathematics of the era. The nature of geometric proofs naturally leads one to think primarily in terms of relative proportions. This now-foreign assumption, that the answer to a problem isn't a number but a ratio, is one of the more subtle impediments to understanding historical scientific publications. A modern reader can follow the proofs without too much trouble, even though a deep familiarity with Euclid and Appolonius is now rare. But without understanding the *why* of the proofs, one can quickly get lost. Building and exploring models based on this worldview is essential to truly understanding the foundations of modern science.

Kepler also represents a critical turning-point in the history of science, as we moved towards a purely mathematical understanding of the universe and away from a syllogistic one. Part of this change was a radical shift in how the beauty of scientific propositions were understood. We still rely on beauty as an important heuristic for evaluating hypotheses, of course. The preference for certain forms of equations or for theories with few special cases is no less of an aesthetic judgment than trying to find the Platonic solids embodied in the structure of the solar system. It wasn't relying on beauty that led Kepler wrong, it was relying on the *wrong kind* of beauty. That was a lesson which took at least another century to learn, but Kepler was a critical part of the change. While he never did fully give up on the *Mysterium Cosmographicum* model, he still spent decades processing Tycho Brahe's observations, slowly disproving his earlier theory. The result, Kepler's laws of planetary motion, was the first model of orbital motion based primarily on data, not a philosophical ideal. This was a crucial step towards Newton's *hypotheses non fingo*.

Planet pair	Real (mean)	Real (aphelion)	Model (mean)	Model (aphelion)
Mercury-Venus	1.846	2.339	2.124	1.776
Venus-Earth	1.389	1.369	1.294	1.306
Earth-Mars	1.520	1.405	1.414	1.522
Mars-Jupiter	3.421	3.583	3.466	3.322
Jupiter-Saturn	1.835	1.823	1.922	1.935
Saturn-Uranus	2.011	2.0312	1.664	1.650
Uranus-Neptune	1.567	1.626	1.638	1.579

Table 1: *Ratios of orbital sizes using real world and model data*

Planet pair	Difference (mean)	Difference (aphelion)
Mercury-Venus	0.278	-0.563
Venus-Earth	-0.095	-0.063
Earth-Mars	-0.106	0.116
Mars-Jupiter	0.045	-0.261
Jupiter-Saturn	0.088	0.112
Saturn-Uranus	-0.346	-0.381
Uranus-Neptune	0.070	-0.04

Table 2: *Differences of real world and model ratios*

References

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